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Graph Optimization Problems on a Bethe Lattice

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The p-partitioning and p-coloring problems on a Bethe lattice of coordination number z are analyzed. It is shown that these two NP-complete optimization problems turn out to be equivalent to finding the ground-state energy of p-state Potts models with frustration. Numerical calculation of the cost function of both problems are carried out for several values of z and p. In the case of $p=2$ the results are identical to those obtained by Mézard and Parisi for the case of the bipartitioning problem. A numerical upper bound to the chromatic number is found for several values of z.

KEY WORDS: Graph partitioning problem; Potts spin glass.

1. INTRODUCTION

The equivalence of combinatorial optimization problems and spin-glass models allows us to apply the methods of statistical mechanics to these and possibly to other NP-complete problems.[1] Fu and Anderson[2] have shown that the bipartitioning of a random graph, in the case of extensive connectivity, is equivalent to finding the ground-state energy of the Sherrington and Kirkpatrick spin-glass model.[3] This result has been extended to the case of the partition of a graph in p subsets by Kanter and Sompolinsky.[4] They have shown that the partitioning and coloring of a random graph, in the case of extensive connectivity, are mapped onto the infinite-range p-state Potts glass.[5,6]

The bipartitioning of a random graph, in the case of intensive connectivity, has been considered by Banavar *et al.*,[7] who have shown that the problem is related to the Ising spin glass on a Bethe lattice of coordination number equal to the connectivity. The bipartitioning problem on a Bethe lattice was studied further by Sherrington and Wong[8] and Mézard and

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Parisi.[9] They estimated the cost function, which is related to the ground-state energy of the corresponding spin-glass model, and found results close to the numerical values obtained by Banavar *et al.*[7] The bipartitioning problem with finite connectivity was also studied by Liao,[12] who used the replica method to obtain solutions with cost functions lower than that given by the spin-glass solution.

In this paper I analyze the p-partitioning and p-coloring problems on a Bethe lattice of coordination number z. The study of these problems on such a lattice is motivated by the relationship between an infinite Bethe lattice and an infinite random graph of intensive connectivity. If one considers a random graph with N vertices such each vertex is connected to exactly z other vertices, then, in the $N \to \infty$ limit, the random graph is expected to behave like a Bethe lattice of coordination number z.[7] In this case, the results obtained here would be also appropriate for infinite random graphs of intensive connectivity z.

The graph partitioning (coloring) problem consists in the partition of a set of vertices into p subsets of equal size in such a way that the number of edges connecting vertices of different subsets (the same subset) is minimized. By associating to each vertex a p-state Potts spin variable, these two NP-complete optimization problems turn out to be equivalent to finding the ground-state energy of p-state Potts model with frustration. In the next section I show how this equivalence is obtained by following a derivation presented by Kanter and Sompolinsky.[4]

2. THE PROBLEM

Consider a lattice of N sites each with coordination number z and let us associate to each site i a p-state Potts spin variable $n_i = 1, 2,..., p$. Given a spin configuration $\{n_i\}$, the number of bonds connecting spins in different states (the cost function of the p-partitioning problem) Γ_{pp} is given by

$$\Gamma_{pp} = \sum_{(ij)} (1 - \delta_{n_i n_j}) \tag{1}$$

where the summation runs over all bonds of the lattice. Similarly, the number of bonds connecting spins in the same state (the cost function of the p-coloring problem) Γ_{cp} is given by

$$\Gamma_{cp} = \sum_{(ij)} \delta_{n_i n_j} \tag{2}$$

For each problem the cost function has to be minimized over the spin configurations $\{n_i\}$ that have the same number of spins in each state. That is, the minimization is subject to the global constraint

$$\sum_i \delta_{n_i n} = N/p \tag{3}$$

for $n = 1, 2,..., p$. This constraint introduces frustration into the problem.

If we let the spins interact according to the Potts prescription,[10] the energy $\mathscr{E}$ of the spin configuration $\{n_i\}$ is given by

$$\mathscr{E} = -\varepsilon \sum_{(ij)} \delta_{n_i n_j} + \mathscr{E}_0 \tag{4}$$

where $\mathscr{E}_0$ is an arbitrary constant. Choose it in such a way that $\mathscr{E}$ is written in the form

$$\mathscr{E} = -J_0 \sum_{(ij)} (p\delta_{n_i n_j} - 1) \tag{5}$$

where the coupling J_0 is related to ε by $J_0 = \varepsilon/p$. The energy $\mathscr{E}$ is related to the cost functions Γ_{pp} and Γ_{cp} by

$$\mathscr{E} = J_0[p\Gamma_{pp} - (p-1) N_b] \tag{6}$$

and

$$\mathscr{E} = -J_0(p\Gamma_{cp} - N_b) \tag{7}$$

where N_b is the total number of bonds. The partitioning and coloring problems are therefore equivalent to minimizing the energy of the Potts model with ferromagnetic ($J_0 > 0$) and antiferromagnetic ($J_0 < 0$) couplings, respectively, both subject to the constraint (3).

In the case of a Bethe lattice the partition constraint (3) is simulated by quenched random fields on the boundary spins. The probability distribution of the field at a boundary spin is chosen to be a symmetric distribution. This means that the distribution is invariant under the permutation of any two pairs of the Potts field components. Due to the recursive relation between the effective fields acting on sites belonging to successive generations of the Bethe lattice, the symmetry property propagates to all sites, which ensures the partition constraint (3) to hold among all spins of the lattice.

By writing the Hamiltonian $\mathscr{H}(\{n_i\})$ of the system as

$$\mathscr{H} = -\sum_{(ij)} J_{ij}(p\delta_{n_i n_j} - 1) \tag{8}$$

one can state the partitioning, the coloring, and the Potts spin glass as follows. Let $P(J_{ij})$ be the probability distribution of the coupling J_{ij}, given by

$$P(J_{ij}) = c\delta(J_{ij} - J) + (1-c)\,\delta(J_{ij} + J) \tag{9}$$

with $J > 0$. One has then (a) for $c = 1$, the partitioning problem; (b) for $c = 0$, the coloring problem; and (c) for $c = 1/2$, the symmetric Potts glass. All three cases should have random and symmetric boundary conditions.

It is worth noting that the Potts spin glass with a $\pm J$ distribution of bonds, defined above, cannot be mapped by a gauge transformation into either the ferromagnetic or the antiferromagnetic Potts model with random boundary condition, except the case $p = 2$, which corresponds to the Ising spin glass. In other words, the three models defined in (a), (b), and (c) cannot be mapped into one another when $p > 2$.

3. THE INTEGRAL EQUATION

Consider a Cayley tree of coordination number z and let $\rho_n^0 = \langle \delta_{n_0 n} \rangle$, $n = 1, 2, \dots, p$, be the thermal average of $\delta_{n_0 n}$ corresponding to the central spin of the tree. It is given by

$$\rho_n^0 = Z_n^0 \Big/ \sum_{l=1}^{p} Z_l^0 \tag{10}$$

with

$$Z_n^0 = \prod_{i=1}^{z} \zeta_n^{0i} \tag{11}$$

where ζ_n^{0i} is obtained recursively by the following hierarchic equation:

$$\zeta_n^{ij} = \sum_{l=1}^{p} e^{p\beta J_{ij}\delta_{nl}} \prod_{k=1}^{K} \zeta_l^{jk} \tag{12}$$

The product in k extends over the $K = z - 1$ spins connected to the site i which belong to the same generation.

From Eqs. (10) and (11) we see that only the ratios, say $\zeta_1^{0i}/\zeta_p^{0i}$, $\zeta_2^{0i}/\zeta_p^{0i}, \dots, \zeta_{p-1}^{0i}/\zeta_p^{0i}$, are relevant to the calculation of ρ_n^0. It is thus more convenient to introduce the $p - 1$ effective field components h_n^{ij} related to these ratios by

$$\exp(p\beta h_n^{ij}) = \zeta_n^{ij}/\zeta_p^{ij} \tag{13}$$

for $n=1, 2,..., p-1$. If we define the function $f(\xi_1, \xi_2,..., \xi_{p-1}, \xi_p)$ by

$$f(\xi_1,..., \xi_p)=\frac{1}{p\beta}\sum_{l=1}^{p} e^{p\beta\xi_l} \tag{14}$$

then the hierarchic equation can be written as

$$\begin{aligned} h_n^{ij} = {} & f(H_1^j, H_2^j,..., J_{ij}+H_n^j,..., H_{p-1}^j, 0) \\ & - f(H_1^j, H_2^j,..., H_n^j,..., H_{p-1}^j, J_{ij}) \end{aligned} \tag{15}$$

for $n=1, 2,..., p-1$, where

$$H_n^j = \sum_{k=1}^{K} h_n^{jk} \tag{16}$$

for $l=1, 2,..., p-1$.

The vector effective fields $(h_1^{ij}, h_2^{ij},..., h_{p-1}^{ij}) \equiv h^{ij}$ and $(h_1^{jk}, h_2^{jk},..., h_{p-1}^{jk}) \equiv h^{jk}$, $k=1, 2,..., K$, are interpreted as random vector variables. The random vector variables h^{jk}, $k=1, 2,..., K$, are supposed to be independent random vector variables with the same probability distribution. In the limit of an infinite Cayley tree this probability distribution will be identical to the probability distribution of the variable h^{ij}. Denoting this limiting distribution by $g(h_1, h_2,..., h_{p-1}) \equiv g(h)$, we have the following integral equation:

$$\begin{aligned} g(h) = {} & \int dJ'\, P(J') \prod_{l=1}^{p-1} \delta(h_l - f_l + f_p) \\ & \times \prod_{k=1}^{K} g(h^k)\, dh_1^k\, dh_2^k \cdots dh_{p-1}^k \end{aligned} \tag{17}$$

where f_l and f_p are defined by

$$f_l = f(H_1, H_2,..., J'+H_l,..., H_{p-1}, 0) \tag{18}$$

for $l=1, 2,..., p-1$, and

$$f_p = f(H_1, H_2,..., H_{p-1}, J') \tag{19}$$

with

$$H_l = \sum_{k=1}^{K} h_l^k \tag{20}$$

for $l=1, 2,..., p-1$.

Let us introduce the characteristic function $\phi(x_1, x_2, \dots, x_{p-1}) \equiv \phi(x)$ defined by

$$\phi(x) = \int \exp\left(i \sum_{l=1}^{p-1} x_l h_l\right) g(h)\, dh_1\, dh_2 \cdots dh_{p-1} \tag{21}$$

so that

$$g(h) = (2\pi)^{-p+1} \int \exp\left(-i \sum_{l=1}^{p-1} x_l h_l\right) \phi(x)\, dx_1\, dx_2 \cdots dx_{p-1} \tag{22}$$

From Eq. (17) we obtain the following integral equation for $\phi(x)$:

$$\phi(x) = \int dJ'\, P(J') \exp\left[i \sum_{l=1}^{p-1} x_l(f_l - f_p)\right] G(H)\, dH_1\, dH_2 \cdots dH_{p-1} \tag{23}$$

where $G(H) \equiv G(H_1, H_2, \dots, H_{p-1})$ is given by

$$G(H) = (2\pi)^{-p+1} \int \exp\left(-i \sum_{l=1}^{p-1} y_l H_l\right) [\phi(y)]^K\, dy_1\, dy_2 \cdots dy_{p-1} \tag{24}$$

and f_l and f_p by expressions (18) and (19).

In order to exploit the full symmetry of the model, it is convenient to introduce a function $\tilde{\phi}(x_1, x_2, \dots, x_{p-1}, x_p) \equiv \tilde{\phi}(x)$ of p variables. Then the solution $\phi(x)$ of the integral equation (23) is of the form of the rhs of the equation

$$\phi(x_1, x_2, \dots, x_{p-1}) = \tilde{\phi}(x_1, x_2, \dots, x_{p-1}, -x_1 - x_2 \cdots - x_{p-1}) \tag{25}$$

Indeed, if we define $\tilde{G}(\xi_1, \xi_2, \dots, \xi_{p-1}, \xi_p) \equiv \tilde{G}(\xi)$ by

$$\tilde{G}(\xi) = (2\pi)^{-p} \int \exp\left(-i \sum_{l=1}^{p} y_l \xi_l\right) [\tilde{\phi}(y)]^K\, dy_1\, dy_2 \cdots dy_p \tag{26}$$

we have

$$\phi(x) = \int dJ'\, P(J') \exp\left(i \sum_{l=1}^{p} x_l \tilde{f}_l\right) \tilde{G}(\xi)\, d\xi_1\, d\xi_2 \cdots d\xi_p \tag{27}$$

where

$$\tilde{f}_l = f(\xi_1, \xi_2, \dots, J' + \xi_l, \dots, \xi_{p-1}, \xi_p) \tag{28}$$

for $l = 1, 2, \dots, p$, and on the rhs of Eq. (27) $x_p = -x_1 - x_2 \cdots - x_{p-1}$, so that $\phi(x)$ is of the form of the rhs of Eq. (25).

At the boundary, the probability distribution has to be symmetric in order to ensure that the constraint (3) is satisfied on the average. This implies that we should look for symmetric solutions for $\tilde{\phi}(x)$, that is, such that

$$\tilde{\phi}(x_1,..., x_l,..., x_n,..., x_p) = \tilde{\phi}(x_1,..., x_n,..., x_l,..., x_p)$$

for any permutation.

4. THE ZERO-TEMPERATURE LIMIT

In the limit of zero temperature, $\beta \to \infty$, the function $f(\xi)$ reduces to

$$f(\xi) = \max(\xi_1, \xi_2,..., \xi_p) \tag{29}$$

In this limit I show that the solution of the integral equation (23) is of the form

$$\tilde{\phi}(x) = \sum_{\{\sigma_j\}}{}' A(\sigma) \exp\left(iJ \sum_{l=1}^{p} \sigma_l x_l\right) \tag{30}$$

where $\sigma_j = 0, 1$, and $A(\sigma) \equiv A(\sigma_1, \sigma_2,..., \sigma_p)$ are coefficients to be determined. The term corresponding to $\sigma = 0$ is excluded from the summation.

First, we have

$$[\tilde{\phi}(x)]^K = \sum_{\{\tau_j\}} B(\tau) \exp\left(iJ \sum_{l=1}^{p} \tau_l x_l\right) \tag{31}$$

where $\tau_j = 0, 1, 2,..., K$, and $B(\tau) \equiv B(\tau_1, \tau_2,..., \tau_p)$ are related to $A(\sigma)$ by

$$B(\tau) = \sum_{\{\sigma_j^1\}} \sum_{\{\sigma_j^2\}} \cdots \sum_{\{\sigma_j^K\}}{}' A(\sigma^1) A(\sigma^2) \cdots A(\sigma^K) \tag{32}$$

where the summation has the restriction $\tau = \sigma^1 + \sigma^2 + \cdots + \sigma^K$. Therefore

$$\tilde{G}(\xi) = \sum_{\{\tau_j\}} B(\tau) \prod_{l=1}^{p} \delta(\xi_l - J\tau_l) \tag{33}$$

After substituting this in Eq. (27) and taking into account Eq. (29) we get

$$\begin{aligned}\phi(x) = c \sum_{\{\tau_j\}} B(\tau) \exp\left[iJ \sum_{l=1}^{p} x_l \max(\tau_1,..., \tau_l + 1,..., \tau_p)\right] \\ + (1-c) \sum_{\{\tau_j\}} B(\tau) \exp\left[iJ \sum_{l=1}^{p} x_l \max(\tau_1,..., \tau_l - 1,..., \tau_p)\right]\end{aligned} \tag{34}$$

where, of course, $x_p = -x_1 - x_2 \cdots - x_{p-1}$.

Let us define $\alpha_l^+(\tau)$ and $\alpha_l^-(\tau)$ by

$$\alpha_l^+ = \max(\tau_1,..., \tau_l + 1,..., \tau_p) - \max(\tau_1,..., \tau_l,..., \tau_p) \tag{35}$$

and

$$\alpha_l^- = \max(\tau_1,..., \tau_l - 1,..., \tau_p) - \max(\tau_1,..., \tau_l,..., \tau_p) + 1 \tag{36}$$

It is easy to check that if $\tau_l \geqslant \tau_n$, then $\alpha_l^+ = 1$, otherwise $\alpha_l^+ = 0$; and that if $\tau_l > \tau_n$, then $\alpha_l^- = 0$, otherwise $\alpha_l^- = 1$. By using the property $x_1 + x_2 + \cdots + x_p = 0$ we get

$$\begin{aligned}\phi(x) = c \sum_{\{\tau_j\}} B(\tau) \exp\left(iJ \sum_{l=1}^{p} \alpha_l^+ x_l\right)\\ + (1-c) \sum_{\{\tau_j\}} B(\tau) \exp\left(iJ \sum_{l=1}^{p} \alpha_l^- x_l\right)\end{aligned} \tag{37}$$

which is of the form of the rhs of Eq. (30), since $\alpha_l^\pm$ takes only the values 0 and 1. By comparing the rhs of Eq. (30) with Eq. (37), we obtain

$$A(\sigma) = cA^+(\sigma) + (1-c) A^-(\sigma) \tag{38}$$

where

$$A^\pm(\sigma) = \sum_{\{\tau_j\}:\, \sigma = \alpha^\pm(\tau)} B(\tau) \tag{39}$$

with $\alpha^\pm = (\alpha_1^\pm, \alpha_2^\pm,..., \alpha_p^\pm)$. Equations (38) and (32) constitute a set of equations for the coefficients $A(\sigma)$. Since we are looking for symmetric solutions, $A(\sigma)$ and $B(\tau)$ are chosen to be symmetric. In this case the number of independent A coefficients is p, but if we take into account the normalization condition, this number is reduced to $p-1$.

In order to write explicit expressions for $A(\sigma)$ and $A^\pm(\sigma)$, we introduce the notation A_l and $A_l^\pm$ defined by $A_l = A(\sigma_1, \sigma_2,..., \sigma_p)$ and $A_l^\pm = A^\pm(\sigma_1, \sigma_2,..., \sigma_p)$ with $l = \sigma_1 + \sigma_2 + \cdots + \sigma_p$. We have then

$$A_l = cA_l^+ + (1-c) A_l^- \tag{40}$$

and

$$A_0^+ = 0 \tag{41a}$$

$$A_l^+ = \sum_{\tau_l=1}^{K} \sum_{\tau_{l+1}=0}^{\tau_l - 1} \cdots \sum_{\tau_p=0}^{\tau_l - 1} B(\tau_l, \tau_l,..., \tau_l, \tau_{l+1},..., \tau_p) \tag{41b}$$

for $l = 1, 2, ..., p-1$,

$$A_p^+ = \sum_{\tau_p = 0}^{K} B(\tau_p, \tau_p, ..., \tau_p) \tag{41c}$$

and

$$A_l^- = 0 \tag{42a}$$

for $l = 0, 1, 2, ..., p-2$, and

$$A_{p-1}^- = A_1^+ \tag{42b}$$

$$A_p^- = 1 - pA_1^+ \tag{42c}$$

5. FREE ENERGY

The free energy per site F can be calculated from

$$F = \frac{z}{2} \int F_2(\xi, \xi') \tilde{G}(\xi) \tilde{G}(\xi') \, d\xi_1 \cdots d\xi_p \, d\xi'_1 \cdots d\xi'_p$$

$$-(z-1) \int F_1(\xi) \tilde{G}_1(\xi) \, d\xi_1 \cdots d\xi_p \tag{43}$$

where $\tilde{G}(\xi)$ is given by Eq. (26) and $\tilde{G}_1(\xi)$ by

$$\tilde{G}_1(\xi) = (2\pi)^{-p} \int \exp\left(-i \sum_{l=i}^{p} y_l \xi_l \right) [\tilde{\phi}(y)]^z \, dy_1 \cdots dy_p \tag{44}$$

The functions $F_1(\xi)$ and $F_2(\xi, \xi')$ are given by

$$F_1(\xi) = -\beta^{-1} \ln \sum_{l=1}^{p} \exp(p\beta\xi_l) \tag{45}$$

$$F_2(\xi, \xi') = -\beta^{-1} \int dJ' \, P(J') \ln \sum_{l,l'=1}^{p} \exp[\beta J'(p\delta_{ll'} - 1) + p\beta(\xi_l + \xi'_{l'})] \tag{46}$$

If the rhs of Eq. (43) is thought of as a function of $g(h)$ through Eqs. (44), (26), and (22), then one can prove that $g(h)$ satisfying Eq. (17) makes this functional stationary under the restriction

$$\int g(h) \, dh_1 \, dh_2 \cdots dh_{p-1} = 1 \tag{47}$$

In the limit of zero temperature, the energy per site E is obtained from $E = \lim F$ when $\beta \to \infty$. By using the solution of the integral equation given by Eq. (30), we get

$$E = \frac{z}{2}[cE_2^+ + (1-c)\, E_2^-] - (z-1)\, E_1 \tag{48}$$

where

$$E_1 = -pJ \sum_{\{\tau_j\}} \sum_{\{\sigma_j\}} A(\sigma)\, B(\tau) \max(\tau_1 + \sigma_1, \tau_2 + \sigma_2, ..., \tau_p + \sigma_p) \tag{49}$$

and

$$E_2^{\pm} = -J \sum_{\{\tau_j\}} \sum_{\{\tau_j'\}} B(\tau)\, B(\tau') \max(\omega_{11}^{\pm}, \omega_{12}^{\pm}, ..., \omega_{pp}^{\pm}) \tag{50}$$

with

$$\omega_{ll'}^{\pm} = \pm(p\delta_{ll'} - 1) + p(\tau_l + \tau_{l'}') \tag{51}$$

From the energy per site E we can get the cost function per site γ_{pp} for the partitioning problem by using Eq. (6) with $J_0 = J$. We have

$$\gamma_{pp} = \frac{1}{p}\left[\frac{E}{J} + \frac{z}{2}(p-1)\right] \tag{52}$$

Analogously, we get the cost function per site γ_{cp} for the coloring problem by using Eq. (7) with $J_0 = -J$. It is given by

$$\gamma_{cp} = \frac{1}{p}\left(\frac{E}{J} + \frac{z}{2}\right) \tag{53}$$

The equations for the coefficients A always admit the trivial solution (the paramagnetic solution) given by $A(\sigma) = 1$ if $\sigma = (1, 1, 1, ..., 1)$ and $A(\sigma) = 0$, otherwise, from which we get $B(\tau) = 1$ if $\tau = (K, K, K, ..., K)$ and $B(\tau) = 0$, otherwise. This solution gives $E = -Jz[c(p-1) + 1 - c]/2$, so that $\gamma_{pp} = 0$ and $\gamma_{cp} = 0$, that is, the trivial solution corresponds to zero cost function for both problems. For further use, let us define the quantity γ by

$$\gamma = \frac{1}{p}\left\{\frac{E}{J} + \frac{z}{2}[c(p-1) + (1-c)]\right\} \tag{54}$$

which gives the cost function per site for the p-partitioning problem when $c = 1$, and for the p-coloring problem when $c = 0$, and is essentially the difference in energy between a certain solution and the paramagnetic solution.

6. THE CASE $p=2$

When $p=2$, Eqs. (42) give $A_l^+ = A_l^-$, $l=0, 1, 2$, so that the rhs of Eq. (40) is independent of c. We have also $E_2^+ = E_2^-$, so that E and γ are also independent of c. These results merely reflect the fact that the three problems, the bipartitioning, the bicoloring, and the Ising spin glass, are equivalent; one can transform one onto another by a gauge transformation. The three coefficients A_0, A_1, and A_2 to be found are not independent since $A_0=0$ and $2A_1 + A_2 = 1$, due to the normalization condition. Since $A_2 = A_2^+$, we obtain from expressions (41c) and (32) the following equation for A_2:

$$A_2 = \sum_{l=0}^{[K/2]} \binom{K}{2l} \binom{2l}{l} A_2^{K-2l} A_1^{2l} \tag{55}$$

where $[x]$ is the integer part of x and $A_1 = (1 - A_2)/2$. This equation was obtained by Sherrington and Wong[8] and Mézard and Parisi.[9] According to Mézard and Parisi,[9] the coefficient A_2 should be interpreted as the number of idle spins.

From the solutions of Eq. (55) we calculate the energy per site E and the cost function per site γ. Table I shows the numerical results for several values of the coordination number z. All results, including the energy, are identical to those obtained by Mézard and Parisi[9] for the bipartitioning problem.

Table I. The Coefficients A_1 and A_2, for the Energy per Site E and the Cost Function per Site γ, for Several Values of z, for the Case $p=2$

z	A_1	A_2	$-E$	γ
3	0.333333	0.333333	1.277778	0.111111
4	0.400000	0.200000	1.488000	0.256000
5	0.385603	0.228793	1.691329	0.404335
6	0.416453	0.167093	1.855251	0.572374
7	0.408726	0.182547	2.021556	0.739222
8	0.426902	0.146196	2.162203	0.918899
9	0.422281	0.155438	2.305790	1.097105
10	0.434291	0.131418	2.431401	1.284299
11	0.431363	0.137274	2.559379	1.470311
12	0.439868	0.120263	2.674115	1.662942
15	0.442998	0.114003	3.004463	2.247768
20	0.453387	0.093225	3.482165	3.258918

7. THE CASE $p = 3$

In the case $p = 3$ we have to find the coefficients A_0, A_1, A_2, A_3. Since $A_0 = 0$ and $3A_1 + 3A_2 + A_3 = 1$, due to the normalization condition, there are only, two independent coefficients to be found. I have solved numerically the equations that determine the coefficients A for several values of the coordination number z. From these coefficients I have obtained the coefficients B, and from both of them I have calculated the energy per site E and the cost function per site γ. Besides the paramagnetic solution which is always present and is given by $A_1 = A_2 = 0$ and $A_3 = 1$, I found a spin-glass solution. When both solution were present, I chose the one with higher energy, as is usual in spin-glass problems. Since $\gamma = 0$ for the paramagnetic solution, the spin-glass solution is chosen whenever it gives $\gamma > 0$.

Figure 1 shows γ as a function of c for the case $z = 3$. At $c = c^* = 0.431$ a first-order transition takes place and the system passes from a paramagnetic phase, $0 \leqslant c \leqslant c^*$, to a spin-glass phase, $c^* \leqslant c \leqslant 1$. This behavior is also observed for $z = 4$, as shown in Fig. 2, and for $z = 5$ and $z = 6$. For $z \geqslant 7$, there is no transition and the system remains, for any value of c, in the spin-glass phase. Notice that γ is a convex function of c.

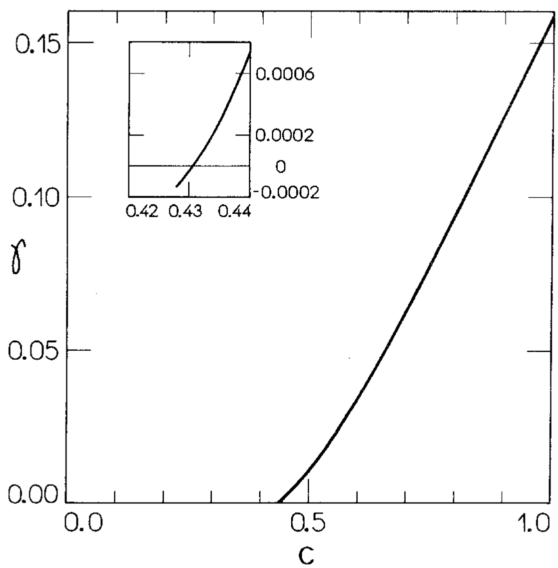

Fig. 1. Cost function γ versus c for $p = 3$ and $z = 3$. A first-order phase transition occurs at $c = c^* = 0.431$. The paramagnetic phase corresponds to $\gamma = 0$. The inset shows an enlargement around the transition point.

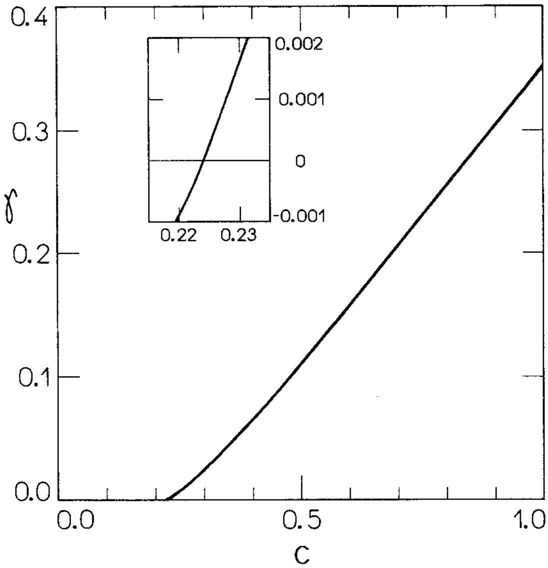

Fig. 2. Cost function γ versus c for $p=3$ and $z=4$. A first-order phase transition occurs at $c=c^*=0.224$. The paramagnetic phase corresponds to $\gamma=0$. The inset shows an enlargement around the transition point.

Tables II, III, and IV display the values of the coefficients A_1, A_2, and A_3, as well as the cost function per site γ for several values of z, for the partitioning problem, the coloring problem, and the symmetric Potts glass, respectively. I show only the spin-glass solution, except for the case $c=0$ and $z=3$, 4, and 5, where only the paramagnetic solution exists.

Table II. The Coefficients A_1, A_2, and A_3, and the Cost Function per Site γ, for Several Values of z, for the Partitioning Problem ($c=1$) in $p=3$ Parts

z	A_1	A_2	A_3	γ
3	0.182277	0.106296	0.134283	0.157800
4	0.122169	0.077529	0.100905	0.354392
5	0.225838	0.089222	0.054819	0.567785
6	0.237821	0.076535	0.056931	0.792304
7	0.247720	0.070614	0.044999	1.028172
8	0.250774	0.070854	0.035114	1.269960
9	0.256474	0.065007	0.035557	1.517360
10	0.260994	0.062592	0.029243	1.770467
11	0.263274	0.061327	0.026197	2.026939

Table III. The Coefficients A_1, A_2, and A_3, and the Cost Function per Site γ, for Several Values of z, for the Coloring Problem ($c=0$) in $p=3$ Colors

z	A_1	A_2	A_3	γ
3	0	0	1	0
4	0	0	1	0
5	0	0	1	0
6	0	0.185080	0.444759	−0.003770
7	0	0.205538	0.383386	0.057602
8	0	0.220417	0.338749	0.128909
9	0	0.230219	0.309342	0.207621
10	0	0.237275	0.288176	0.291550
11	0	0.243800	0.268601	0.380201
12	0	0.248792	0.253625	0.472723

8. THE COLORING PROBLEM

For the case of the coloring problem, $c=0$, there is a drastic simplification of the equations that give the A coefficients. From Eqs. (40) and (42) we see that $A_l=0$ for $l=0, 1, 2,..., p-2$, so that the only surviving coefficients are A_p and A_{p-1}. However, they are not independent, due to the normalization condition $pA_{p-1}+A_p=1$. After some algebra, we obtain the following equation for A_p:

$$A_p=\sum_{N=0}^{K}\binom{K}{N}\mathscr{C}_{Np}A_p^{K-N}A_{p-1}^{N} \tag{56}$$

Table IV. The Coefficients A_1, A_2, and A_3, and the Cost Function per Site γ, for Several Values of z, for the Symmetric $p=3$ State Potts Glass ($c=1/2$)

z	A_1	A_2	A_3	γ
3	0.058409	0.126064	0.446581	0.010235
4	0.093696	0.147672	0.275897	0.111218
5	0.104786	0.153150	0.226192	0.238320
6	0.111742	0.156083	0.196523	0.379357
7	0.116702	0.157982	0.175948	0.530179
8	0.120493	0.159295	0.160635	0.688457
9	0.123521	0.160267	0.148636	0.852689
10	0.126014	0.161013	0.138920	1.021827
11	0.128048	0.161543	0.131226	1.195104

where $A_{p-1} = (1 - A_p)/p$ and $\mathscr{C}_{Np}$ is given by

$$\mathscr{C}_{Np} = p^N - p \sum_{\nu_1 \geqslant 0} \sum_{\nu_2 \geqslant \nu_1 + 1} \cdots \sum_{\nu_p \geqslant \nu_1 + 1} \frac{N!}{\nu_1 !\, \nu_2 ! \cdots \nu_p !} \tag{57}$$

when $N \geqslant p - 1$, where the summations are performed with the restriction $\nu_1 + \nu_2 + \cdots + \nu_p = N$. When $0 \leqslant N < p - 1$, $\mathscr{C}_{Np} = p^N$.

Table V. Values of A_p for Several Values of z for the Coloring Problem in p Colors, for $2 \leqslant p \leqslant 8$

z	A_2	A_3	A_4	A_5	A_6	A_7	A_8
3	0.3333	1.	1.	1.	1.	1.	1.
4	0.2000	1.	1.	1.	1.	1.	1.
5	0.2288	1.	1.	1.	1.	1.	1.
6	0.1671	0.4448	1.	1.	1.	1.	1.
7	0.1825	0.3834	1.	1.	1.	1.	1.
8	0.1462	0.3387	1.	1.	1.	1.	1.
9	0.1554	0.3093	1.	1.	1.	1.	1.
10	0.1314	0.2882	1.	1.	1.	1.	1.
11	0.1373	0.2686	0.4584	1.	1.	1.	1.
12	0.1203	0.2536	0.4183	1.	1.	1.	1.
13	0.1241	0.2414	0.3871	1.	1.	1.	1.
14	0.1115	0.2297	0.3622	1.	1.	1.	1.
15	0.1140	0.2201	0.3431	1.	1.	1.	1.
16	0.1043	0.2118	0.3271	0.5037	1.	1.	1.
17	0.1060	0.2039	0.3127	0.4622	1.	1.	1.
18	0.0983	0.1970	0.3000	0.4346	1.	1.	1.
19	0.0994	0.1909	0.2891	0.4113	1.	1.	1.
20	0.0932	0.1850	0.2794	0.3916	1.	1.	1.
21	0.0939	0.1798	0.2704	0.3754	1.	1.	1.
22	0.0888	0.1751	0.2622	0.3616	0.5186	1.	1.
23	0.0892	0.1706	0.2548	0.3494	0.4830	1.	1.
24	0.0850	0.1664	0.2480	0.3382	0.4594	1.	1.
25	0.0852	0.1626	0.2417	0.3280	0.4393	1.	1.
26	0.0816	0.1590	0.2358	0.3188	0.4216	1.	1.
27	0.0816	0.1556	0.2303	0.3104	0.4063	1.	1.
28	0.0786	0.1525	0.2253	0.3027	0.3932	1.	1.
29	0.0785	0.1495	0.2205	0.2955	0.3817	0.5119	1.
30	0.0758	0.1466	0.2160	0.2887	0.3714	0.4878	1.
31	0.0757	0.1440	0.2118	0.2824	0.3619	0.4689	1.
32	0.0734	0.1414	0.2078	0.2766	0.3530	0.4522	1.
33	0.0731	0.1390	0.2040	0.2711	0.3447	0.4372	1.
34	0.0711	0.1367	0.2004	0.2659	0.3371	0.4238	1.
35	0.0708	0.1346	0.1970	0.2609	0.3300	0.4121	1.
36	0.0691	0.1325	0.1938	0.2563	0.3234	0.4017	0.5213
37	0.0688	0.1305	0.1907	0.2519	0.3172	0.3924	0.5001

For the particular case of $p=2$ we obtain $\mathscr{C}_{N2}=0$ for N odd and $\mathscr{C}_{N2}=\binom{N}{N/2}$ for N even, from which follows Eq. (55).

For each value of the coordination number z and number of colors p I have obtained A_p by repeated iterations of Eq. (56). Table V displays the results for several values of z and p. When only the paramagnetic solution ($A_{p-1}=0$ and $A_p=1$) is present, then this solution is shown. In this case the cost function is zero.

Let us denote by p^* the chromatic number for a given z, that is, the smaller number of color p for which $\gamma=0$. From Table V we can have numerical upper bounds for p^* for each value of z. We have

$$\begin{array}{lll} p^* \leqslant 3 & \text{for} & 3 \leqslant z \leqslant 5 \\ p^* \leqslant 4 & \text{for} & 6 \leqslant z \leqslant 10 \\ p^* \leqslant 5 & \text{for} & 11 \leqslant z \leqslant 15 \\ p^* \leqslant 6 & \text{for} & 16 \leqslant z \leqslant 21 \\ p^* \leqslant 7 & \text{for} & 22 \leqslant z \leqslant 28 \\ p^* \leqslant 8 & \text{for} & 29 \leqslant z \leqslant 35 \end{array}$$

From the results of Sections 6 and 7 we have, actually, $p^*=3$ for $3 \leqslant z \leqslant 6$.

9. CONCLUSION

I have analyzed the p-partitioning and p-coloring problems on a Bethe lattice of coordination number z for several values of p and z. The study of these problems on such a lattice is motivated by the relationship between an infinite Bethe lattice and an infinite random graph of intensive connectivity. Consider a random graph with N vertices such that each vertex is connected to exactly α other vertices, with α independent of N. According to Banavar *et al.*,[7] in the limit of $N \to \infty$, the probability of small loops decreases as $1/N$, so that the random graph will behave like a Bethe lattice of coordination number α. Therefore, the present results should be compared with those valid for random graphs of intensive connectivity that have an infinite number of vertices. For instance, the results I have obtained for the chromatic number are in good agreement with those given by Lai and Goldschmidt.[11] In the case of $p=2$ the present results are identical to those obtained by Mézard and Parisi[9] for the bipartitioning problem.

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